Step 2. Classify each constrained critical point as a local minimum, local maximum, or saddle point by applying the second derivative test for constrained extrema.

$$d_3 = \begin{vmatrix} 0 & -1 & -3 \\ -1 & 0 & 1 \\ -3 & 1 & 0 \end{vmatrix} = 6 \qquad -d_3 = -6 < 0$$

$\Rightarrow f$ has a constrained local max. at $(7,1)$

Example 2. Use the Lagrange multiplier method to find the local optima of

$$\begin{aligned} \text{minimize/maximize} \quad & \underbrace{x_1^2 + x_2^2 + x_3^2}_{f(x_1,x_2,x_3)} \\ \text{subject to} \quad & \underbrace{2x_1 + x_2 + 4x_3}_{g(x_1,x_2,x_3)} = \underbrace{168}_{c} \end{aligned}$$

$$L(\lambda, x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - \lambda\left[2x_1 + x_2 + 4x_3 - 168\right]$$

$$\nabla L(\lambda, x_1, x_2, x_3) = \begin{bmatrix} -2x_1 - x_2 - 4x_3 + 168 \\ 2x_1 - 2\lambda \\ 2x_2 - \lambda \\ 2x_3 - 4\lambda \end{bmatrix}$$

$$H_L(\lambda, x_1, x_2, x_3) = \begin{bmatrix} 0 & -2 & -1 & -4 \\ -2 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -4 & 0 & 0 & 2 \end{bmatrix}$$

<u>CCPs</u>:

$2x_1 + x_2 + 4x_3 = 168$ ①

$2x_1 = 2\lambda$ ②

$2x_2 = \lambda$ ③

$2x_3 = 4\lambda$ ④

Sub ②,③,④ into ①:

$2\lambda + \frac{\lambda}{2} + 8\lambda = 168$

$\Rightarrow \lambda = 16$

Back into ②,③,④:

$x_1 = 16, \quad x_2 = 8, \quad x_3 = 32$

CCPs: $(16, 16, 8, 32)$

Ex 2 cont.

2^{nd} deriv test: $H(16, 16, 8, 32) = \begin{bmatrix} 0 & -2 & -1 & -4 \\ -2 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -4 & 0 & 0 & 2 \end{bmatrix}$

$$d_3 = \begin{vmatrix} 0 & -2 & -1 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = -10$$

$$d_4 = -(-4)\begin{vmatrix} -2 & -1 & -4 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & -2 & -1 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 4(-16) + 2(-10) = -84$$

Since $d_3 < 0$, $d_4 < 0 \Rightarrow f$ has a local minimum at $(16, 8, 32)$.